A Study On (I,J) – (Gs)* Closed Sets In Bitopological Spaces

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Abstract—In this dissertation, we introduce new class of “(i, j) – (gs)*-closed sets in Bitopological spaces”. Properties of these sets are investigated and we introduce four new spaces namely, (i, j) – T_{(gs)*} spaces, (i, j) - \( \tau \) T_{(gs)*} spaces, (i, j) - \( \tau \)\( \psi \) T_{(gs)*} spaces, (i, j) - \( \psi \) T_{(gs)*} spaces.

Key words: (i, j) – (gs)*-closed sets, (i, j) – T_{(gs)*}, (i, j) - \( \tau \) T_{(gs)*} s, (i, j) - \( \tau \)\( \psi \) T_{(gs)*}, (i, j) - \( \psi \) T_{(gs)*} spaces.

1. INTRODUCTION


In this paper we study the relationships of (i, j) - (gs)*-closed sets. We also obtain the basic properties of (i, j) - (gs)*-closed sets.

Norman Levine [14], Bhattacharya and Lahiri [4] and R.Devi et. al.[8] introduced T_{1/2} spaces, T_{3} and T_{5} spaces respectively.

We now introduce and study four new classes of spaces, namely the class of (i, j) – T_{(gs)*}, spaces, (i, j) – \( \tau \) T_{(gs)*} spaces, (i, j) - \( \tau \)\( \psi \) T_{(gs)*} spaces. We also study some relationships of these spaces with (i, j) - T_{1/2} spaces, (i, j) - T_{3} space and (i, j) – T_{5} spaces.

2. PRELIMINARIES

Definition 2.1: A subset A of a topological space (X, \( \tau \)) is called

1. a semi-open set if A \( \subseteq \) cl (int (A)) and a semi-closed set if int (cl (A)) \( \subseteq \) A.
2. a pre-open set if A \( \subseteq \) Int(cl(A)) and a pre-closed set if cl(int(A)) \( \subseteq \) A.
3. an \( \alpha \)-open set if A \( \subseteq \) int(cl(int(A))) and an \( \alpha \)-closed set if cl(int(int(A))) \( \subseteq \) A.
4. a generalized closed set (briefly g-closed) if cl (A) \( \subseteq \) U whenever A \( \subseteq \) U and U is open in (X, \( \tau \)).
5. a generalized semi-closed set (briefly gs-closed) if scl(A) \( \subseteq \) U whenever A \( \subseteq \) U and U is open in (X, \( \tau \)).
6. a generalized \( \alpha \)-closed set (briefly g\( \alpha \) - closed) if \( \alpha \)cl(A) \( \subseteq \) U whenever A \( \subseteq \) U and U is \( \alpha \)-open in (X, \( \tau \)).
7. a \( \alpha \) -generalized closed set (briefly g\( \alpha \) g-closed) if \( \alpha \)cl(A) \( \subseteq \) U whenever A \( \subseteq \) U and U is open in (X, \( \tau \)).
8. a generalized semi-pre closed set (briefly gsp-closed) if spcl (A) \( \subseteq \) U whenever A \( \subseteq \) U and U is regular open in (X, \( \tau \)).
9. a generalized pre closed set (briefly gp-closed) if pcl(A) \( \subseteq \) U whenever A \( \subseteq \) U and U is open in (X, \( \tau \)).
10. a generalized star closed set (briefly g* -closed) if cl (A) \( \subseteq \) U whenever A \( \subseteq \) U and U is g-open in (X, \( \tau \)).
11. a \( \psi \)-closed if scl(A) \( \subseteq \) U whenever A \( \subseteq \) U and U is g-open in (X, \( \tau \)).
12. a \( \psi \)* -closed if scl(A) \( \subseteq \) U whenever A \( \subseteq \) U and U is g-open in (X, \( \tau \)).
13. a g**-closed if cl(A) \( \subseteq \) U whenever A \( \subseteq \) U and U is g**-open in (X, \( \tau \)).
14. a \( g\alpha \)-closed if cl(A) \( \subseteq \) U whenever A \( \subseteq \) U and U is \( \alpha \)-open in (X, \( \tau \)).
15. a \( g\alpha \) semi-closed if cl(A) \( \subseteq \) U whenever A \( \subseteq \) U and U is semi-open in (X, \( \tau \)).

If A is a subset of X with topology \( \tau \), then the closure of A is denoted by \( \tau \)- cl(A) or cl(A). The interior of A is denoted by \( \tau \)- int(A) and the complement of A in X is denoted by A\( ^{c} \).

Definition 2.2: A subset A of a bitopological space (X, \( \tau _{1}, \tau _{j} \)) is called

1. a (i,j)-g-closed if \( \tau _{1} \)- cl(A) \( \subseteq \) U whenever A \( \subseteq \) U and U is open in \( \tau _{1} \).
2. a (i,j)-o-closed if \( \tau _{1} \)- cl(A) \( \subseteq \) U whenever A \( \subseteq \) U and U is semi-open in \( \tau _{1} \).
3. a (i,j)-g\( \alpha \) -closed if \( \tau _{1} \)- cl(A) \( \subseteq \) U whenever A \( \subseteq \) U and U is g-open in \( \tau _{1} \).
4. a (i,j)-g\( \alpha \) semi-closed if \( \tau _{1} \)- scl(A) \( \subseteq \) U whenever A \( \subseteq \) U and U is semi-open in \( \tau _{1} \).
5. a (i,j)-g\( \alpha \) closed if \( \tau _{1} \)- scl(A) \( \subseteq \) U whenever A \( \subseteq \) U and U is open in \( \tau _{1} \).
6. a (i,j)-g\( \alpha \) semi-closed if \( \tau _{1} \)- cl(A) \( \subseteq \) U whenever A \( \subseteq \) U and U is semi-open in \( \tau _{1} \).

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Definition 2.3: A bitopological space \((X, \tau_1, \tau_2)\) is called a
\((i, j)\)-\(T_{1/2}\) space if every \((i,j)\)-g-closed set in it is \(\tau_1\) -
closed.

1. \((i,j)\)-\(T_{1/2}\) space if every \((i, j)\)-g-closed set in it is \(\tau_1\) -
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closed.

3. \textbf{RELATIONSHIPS OF \((i, j)\)-(gs)\*- CLOSED SETS WITH SOME OTHER}

\textbf{CLOSED SETS}

We introduce the following definition.

Definition 3.1: A subset \(A\) of a bitopological space \((X, \tau_1, \tau_2)\) is called a
\((i, j)\)-\(g(s)\)*-closed if \(\tau_1 \cap \text{cl}(A) \subseteq U\) whenever \(A 
\subseteq U\) and \(U \text{ is g-open in } \tau_1\).

Remark 3.2: By setting \(\tau_1 \cap \tau_2\) in definition 3.1 a \((i, j)\)-\(g(s)\)*-closed set is a
g(s)* closed set.

Proposition 3.3: Every \(\tau_1\) – closed subset of \((X, \tau_1, \tau_2)\) is (i, j)-\(g(s)\)*-closed.

The following example shows that the converse of the above proposition is not true.

Example 3.4: Let \(X = \{a, b, c\}, \tau_1 = \{\Phi, X, \{a, c\}\}, \\{a, b\}\), \(\tau_2 = \{\Phi, X, \{a\}\}\).
The set \(\{b\}\) is \(\tau_1\) – \(g(s)\)* closed but not \(\tau_2\) closed.

Proposition 3.5: Every \((i, j)-(gs)\)*-closed set is \((i, j)\)-g closed set.

The following example shows that the converse of the above proposition is not true.

Example 3.6: Let \(X = \{a, b, c\}, \tau_1 = \{\Phi, X, \{a, b\}\}, \tau_2 = \{\Phi, X, \{a\}\}\).
The set \(\{c\}\) is \(\tau_1\) – g closed but not \(\tau_2\)-\(g(s)\)*-closed.

Proposition 3.7: Every space \((i, j)-(gs)\)*-closed set is \((i, j)\)-g* -
closed set.

The converse of the above proposition is not true as seen in the following example.

Example 3.8: Let \(X = \{a, b, c\}, \tau_1 = \{\Phi, X, \{a, b\}\}, \tau_2 = \{\Phi, X, \{a\}\}\).
The set \(\{c\}\) is \((1, 2)\) – g closed but not \((1, 2)\)-(\(g(s)\)* closed.

Proposition 3.9: Every \((i,j)-(gs)\)*-closed set is \((i,j)\)-g** -
closed set.

The converse of the above proposition is not true as seen in the following example.

Example 3.10: Let \(X = \{a, b, c\}, \tau_1 = \{\Phi, X, \{a, b\}\}, \tau_2 = \{\Phi, X, \{a\}\}\).
The set \(\{c\}\) is \((1, 2)\) – g** closed but not \((1, 2)\)-(\(g(s)\)* closed.

Proposition 3.11: Every \((i,j)-(gs)\)*-closed set is \((i,j)\)-ag closed set.

The converse of the above proposition is not true as seen in the following example.

Example 3.12: Let \(X = \{a, b, c\}, \tau_1 = \{\Phi, X, \{a, b\}\}, \tau_2 = \{\Phi, X, \{a\}\}\).
The set \(\{b\}\) is \((1, 2)\) – ag closed but not \((1, 2)-(gs)*\) closed.

Proposition 3.13: Every \((i,j)-(gs)*\)-closed set is \((i,j)\)-ga closed set.

The converse of the above proposition is not true as seen in the following example.

Example 3.14: Let \(X = \{a, b, c\}, \tau_1 = \{\Phi, X, \{a, b\}\}, \tau_2 = \{\Phi, X, \{a\}\}\).
The set \(\{c\}\) is \((1, 2)\) – gsp closed but not \((1, 2)-(gs)*\) closed.

Proposition 3.15: Every \((i,j)-(gs)*\)-closed set is \((i,j)\)-gsp closed set.

The converse of the above proposition is not true as seen in the following example.

Example 3.16: Let \(X = \{a, b, c\}, \tau_1 = \{\Phi, X, \{a, b\}\}, \tau_2 = \{\Phi, X, \{a\}\}\).
The set \(\{b\}\) is \((1, 2)\) – gsp closed but not \((1, 2)-(gs)*\) closed.

Proposition 3.17: Every \((i,j)-(gs)*\)-closed set is \((i,j)\)-gsp closed set.

The converse of the above proposition is not true as seen in the following example.

Example 3.18: Let \(X = \{a, b, c\}, \tau_1 = \{\Phi, X, \{a, c\}\}, \tau_2 = \{\Phi, X, \{a\}\}\).
The set \(\{c\}\) is \((1, 2)\) – gsp closed but not \((1, 2)-(gs)*\) closed.

Proposition 3.19: Every \((i,j)-(gs)*\)-closed set is \((i,j)\)-gp closed set.

The converse of the above theorem is not true as seen in the following example.

Example 3.20: Let \(X = \{a, b, c\}, \tau_1 = \{\Phi, X, \{a\}\}, \tau_2 = \{\Phi, X, \{a\}\}\).
The set \(\{c\}\) is \((1, 2)\) – gp closed but not \((1, 2)-(gs)*\) closed.

Proposition 3.21: Every \((i,j)-(gs)*\)-closed set is \((i,j)\)-\(\omega\) closed set.

The converse of the above theorem is not true as seen in the following example.

Example 3.22: Let \(X = \{a, b, c\}, \tau_1 = \{\Phi, X, \{a\}\}, \tau_2 = \{\Phi, X, \{a\}\}\).
The set \(\{c\}\) is \((1, 2)\) – \(\omega\) closed but not \((1, 2)-(gs)*\) closed.

Proposition 3.23: Every \((i,j)-(gs)*\)-closed set is \((i,j)\)-\(\psi\) closed set.

The converse of the above theorem is not true as seen in the following example.

Example 3.24: Let \(X = \{a, b, c\}, \tau_1 = \{\Phi, X, \{a\}\}, \tau_2 = \{\Phi, X, \{a\}\}\).
The set \(\{b\}\) is \((1, 2)\) – \(\psi\) closed but not \((1, 2)-(gs)*\) closed.
Proposition 3.25: Every (i,j)-(gs)*-closed set is (i,j)-\(\psi^*\) closed set.

The converse of the above theorem is not true as seen in the following example.

Example 3.26: Let \(X = \{a, b, c\}\), \(\tau_1 = \{\emptyset, X, \{a\}\}\), \(\tau_2 = \{X, \emptyset, \{a\}, \{b, c\}\}\), \(\tau_3 = \{\emptyset, X, \{a\}\}\). Then the set \(\{b\}\) is \((1, 2) - \psi^*\) closed but not \((1, 2) - (gs)^*\) closed.

Proposition 3.27: Every \((i,j)-(gs)^*\)-closed set is \((i,j) - \alpha^*\) closed set. All the above results can be represented by following diagram.

Where \(A \rightarrow B\) represents \(A\) implies \(B\) and \(A \rightarrow\) \(B\) represents \(A\) does not imply \(B\).

4. Basic properties of \((i, j) - (gs)^*\) closed sets

Proposition 4.1: Union of any two \((i,j)- (gs)^*\) closed sets is again \((i,j)- (gs)^*\) closed.

Proof: Let \(A\) and \(B\) be \((i,j) - (gs)^*\) closed sets. Then \(A \subseteq U\) and \(B \subseteq U\) where \(U\) is \((gs)\) open in \(\tau_i\). This implies \(\tau_i - \text{cl}(A) \subseteq U\) and \(\tau_i - \text{cl}(B) \subseteq U\). Now, \(\tau_i - \text{cl}(A \cup B) = \tau_i - \text{cl}(A) \cup \tau_i - \text{cl}(B) \subseteq U\). Hence \(\tau_i - \text{cl}(A \cup B) \subseteq U\) whenever \((A \cup B) \subseteq U\) and \(U\) is \((gs)\) open in \(\tau_i\). Therefore \(A \cup B\) is \((i,j) - (gs)^*\) closed sets.

Remark 4.2: The intersection of two \((i,j) - (gs)^*\) closed set need not be \((i,j) - (gs)^*\) closed set as seen from the following example.

Example 4.3: Let \(X = \{a, b, c\}\), \(\tau_1 = \{X, \emptyset, \{a\}\}\), \(\tau_2 = \{X, \emptyset, \{b\}, \{c\}, \{b, c\}, \{a, c\}\}\), \(\tau_3 = \{\emptyset, X, \{a\}\}\). Here \(A = \{a, b\}\) and \(B = \{b, c\}\) are \((2, 1) - (gs)^*\) closed sets, but \(A \cap B = \{b\}\) is not a \((2, 1) - (gs)^*\) closed set.

Remark 4.4: \((1, 2) - (gs)^*\) closed is generally not equal to \((2, 1) - (gs)^*\) closed as seen from the following example.

Example 4.5: Let \(X = \{a, b, c\}\), \(\tau_1 = \{X, \emptyset, \{a\}\}\), \(\tau_2 = \{X, \emptyset, \{b\}, \{c\}, \{b, c\}, \{a, c\}\}\), \(\tau_3 = \{\emptyset, X, \{a\}\}\). Here \(A = \{b\}\) is not \((1, 2) - (gs)^*\) closed, but \(A = \{b\} \subseteq (2, 1) - (gs)^*\) closed set.

Example 4.6: If \(A\) is \((i,j)-(gs)^*\) closed and \(\tau_i - (gs)^*\) open, then \(A\) is \(\tau_i - \text{cl}(A)\) closed.

Proof: Let \(A\) be both \((i,j)-(gs)^*\) closed and \(\tau_i - (gs)^*\) open. Then \(\tau_i - \text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \((gs)\)-open in \(\tau_i\). But, since \(A\) is \(\tau_i - (gs)^*\) open and \(A \subseteq A\), \(\tau_i - \text{cl}(A) \subseteq A\). Therefore \(A = \tau_i - \text{cl}(A)\). Hence \(A\) is \(\tau_i - \text{cl}(A)\) closed.

Proposition 4.7: If \(A\) is \((i,j)-(gs)^*\) closed, then \(\tau_j - \text{cl}(A) \\setminus A\) contains no \(\tau_i - (gs)^*\) closed set.

Proof: Let \(A\) be \((i,j)-(gs)^*\) closed sets and let \(F\) be a \(\tau_i - \)
We introduce the following definitions:

Definition 5.3: A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called a $(i, j)$-$(\psi T_{\psi})^*_{[k]}$ closed set, if every $(i, j)$-gs closed set in it is $(i, j)$-$(\psi T_{[\psi]}^*)$ closed.

Definition 5.4: A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called a $(i, j)$-$(\psi T_{\psi})^*_{[\psi]}$ closed, if every $(i, j)$-$\psi^*[\psi]$ closed set in it is $(i, j)$-$(\psi T_{[\psi]}^*)$ closed.

Definition 5.5: Every $(i, j) - T_6$ space is an $(i, j) - (\psi T_{\psi})^*_{[\psi]}$ space, but not conversely.

Example 5.6: Let $X = \{ a, b, c \}$, $\tau_1 = \{ \emptyset, X, \{ a \}, \{ a, b \} \}$, $\tau_2 = \{ \emptyset, X, \{ a \}, \{ b \} \}$. Then $(X, \tau_1, \tau_2)$ is an $(i, j) - (\psi T_{\psi})^*_{[\psi]}$ closed set but not $(i, j) - T_6$ closed.

Therefore $(X, \tau_1, \tau_2)$ is not a $(i, j) - T_6$ space.

Definition 5.7: Every $(i, j) - T_\psi$ space is an $(i, j) - \psi^*[\psi]$-$(\psi T_{\psi})^*_{[\psi]}$ closed space, but not conversely.

Example 5.8: Let $X = \{ a, b, c \}$, $\tau_1 = \{ \emptyset, X, \{ a \}, \{ a, b \} \}$, $\tau_2 = \{ \emptyset, X, \{ b \}, \{ b, c \} \}$. Then $(X, \tau_1, \tau_2)$ is an $(i, j) - \psi^*[\psi]$ closed set but not $(i, j) - T_\psi$ closed.

Therefore $(X, \tau_1, \tau_2)$ is not a $(i, j) - T_\psi$ space.

Proposition 5.9: Every $(i, j) - T_{1/2}$ space is an $(i, j) - \varepsilon T_{\varepsilon}$ $(\psi T_{\psi})^*_{[\psi]}$ space, but not conversely.

Example 5.10: Let $X = \{ a, b, c \}$, $\tau_1 = \{ \emptyset, X, \{ a \}, \{ a, c \} \}$, $\tau_2 = \{ \emptyset, X, \{ a \} \}$. Then $(X, \tau_1, \tau_2)$ is an $(i, j) - \varepsilon T_{\varepsilon}$ $(\psi T_{\psi})^*_{[\psi]}$ space. But $\{ b \}$ is a $(1, 2)$- closed set but not $(i, j) - T_{1/2}$ closed.

Therefore $(X, \tau_1, \tau_2)$ is not a $(i, j) - T_{1/2}$ space.

Proposition 5.11: Every $(i, j) - T_{1/2}$ space is an $(i, j) - T_{(\psi T_{\psi})^*_{[\psi]}}$ space, but not conversely.

Example 5.12: Let $X = \{ a, b, c \}$, $\tau_1 = \{ \emptyset, X, \{ a \}, \{ a, b \} \}$, $\tau_2 = \{ \emptyset, X, \{ a \} \}$. Then $(X, \tau_1, \tau_2)$ is an $(i, j) - T_{(\psi T_{\psi})^*_{[\psi]}}$ space. But $\{ c \}$ is $(1, 2)$- closed but not $(i, j) - T_{1/2}$ closed.

Therefore $(X, \tau_1, \tau_2)$ is not a $(i, j) - T_{1/2}$ space.

Proposition 5.13: Every $(i, j) - T_{\varepsilon}$ space is an $(i, j) - T_{(\psi T_{\psi})^*_{[\psi]}}$ space, but not conversely.

Example 5.14: Let $X = \{ a, b, c \}$, $\tau_1 = \{ \emptyset, X, \{ a \}, \{ a, b \} \}$, $\tau_2 = \{ \emptyset, X, \{ a \} \}$. Then $(X, \tau_1, \tau_2)$ is an $(i, j) - T_{(\psi T_{\psi})^*_{[\psi]}}$ space. But $\{ c \}$ is $(1, 2) - \varepsilon$ closed but not $(i, j) - T_{1/2}$ closed.

Therefore $(X, \tau_1, \tau_2)$ is not a $(i, j) - T_{1/2}$ space.

All the above results can be represented by following diagram.
Where A→B represents A implies B and A ↛ B represents A does not imply B

REFERENCES